

Supersymmetric $\mathbb{Z}_N \times \mathbb{Z}_M$ Orientifolds in 4D with D-Branes at Angles

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Abstract

We construct orientifolds of type IIA string theory. The theory is compactified on a $T^6/\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold. In addition worldsheet parity in combination with a reflection of three compact directions is modded out. Tadpole cancellation requires to add D-6-branes at angles. The resulting four dimensional theories are $\mathcal{N} = 1$ supersymmetric and non-chiral.

1 Introduction

One of the major issues in string theory is to classify consistent theories in especially 3+1 dimensions. Insights into strong coupling regions of string theory provide reasons to hope that apparently different models are actually equivalent and can be mapped onto each other by duality transformations. Often, strong and weak coupling regions are interchanged in this process. Open strings with Dirichlet boundary conditions e.g. provide a perturbative description of solitonic (non-perturbative) objects (D-p-branes) in type II string theories[1]. This observation was crucial for one of the first conjectures about string dualities - the heterotic/type I duality[2]. Since compactifications of the heterotic string are of particular phenomenological interest one expects the same for type I compactifications. Here, one typically starts with a type II theory on an orbifold. In addition, worldsheet parity (possibly combined with discrete targetspace transformations) is modded out. The consistency requirement of modular invariance is replaced by tadpole cancellation conditions. The resulting models are called orientifolds. This kind of construction has been considered already some time ago[3–7]. The first formulation using the modern language of D-branes and Orientifold fixed planes was given in[8]. Subsequently, several models with different numbers of non-compact dimensions and unbroken supersymmetries have been constructed e.g. in[9–17].

In orientifolds the amount of unbroken supersymmetry depends on the orbifold group and the arrangement of D-branes and O-planes needed for consistent compactifications. In[18] it was pointed out that also D-branes intersecting at angles can leave some supersymmetries unbroken. A concrete realization of this observation in orientifold constructions was worked out in [19, 20]. These authors considered type IIB (IIA) string theory on T^4/\mathbb{Z}_N (T^6/\mathbb{Z}_N) orbifolds with $N = 3, 4, 6$. In addition, they gauged worldsheet parity together with the reflection of two (three) directions of the T^4/\mathbb{Z}_N (T^6/\mathbb{Z}_N) orbifold such that this reflection leaves O-planes intersecting at angles fixed. To cancel the corresponding RR-charges, D-branes intersecting at angles need to be added. In the present paper we are going to supplement this class of models by compactifying type IIA on a $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ orbifold together with imposing invariance under worldsheet parity inversion combined with the reflection of three orbifold directions. We will discuss only models with $\mathcal{N} = 1$ supersymmetry in four dimensions. All possible compactifications of this kind yield non-chiral four dimensional models with different gauge groups and matter content.

In the next section we describe general features of the construction. The third section is devoted to a detailed study of the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold. Subsequently, we briefly give results for all other consistent models, *viz.* the $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6 \times \mathbb{Z}_3$ and the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orientifolds. We conclude by summarizing our results. Three appendices provide technical details of the considered models: appendix A is addressed to the computation of loop channel diagrams, appendix B contains tables of massless spectra, and appendix C describes a projective representation used for the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold.

2 General Setup

Throughout the paper we will discuss models with four non-compact dimensions labeled by x^μ , $\mu = 0, \dots, 3$. In addition, there are six compact directions which we describe by three complex coordinates,

$$z^1 = x^4 + ix^5, \quad z^2 = x^6 + ix^7, \quad z^3 = x^8 + ix^9. \quad (1)$$

Each of those coordinates describes a torus T^2 . In addition, points on these tori are identified under rotations

$$\Theta_1 : z^j \rightarrow e^{2\pi i v_j} z^j, \quad \Theta_2 : z^j \rightarrow e^{2\pi i w_j} z^j, \quad (2)$$

where (Θ_1, Θ_2) denotes an element of the orbifold group $\mathbb{Z}_N \times \mathbb{Z}_M$. (The action on the complex conjugated coordinates just follows from the conjugation of (2).) We will discuss type IIA theories on these manifolds. In addition, we gauge the symmetry generated by $\Omega\mathcal{R}$ where Ω reverses worldsheet parity, and \mathcal{R} reflects the imaginary parts of the z^i ,

$$\mathcal{R} : z^i \rightarrow \bar{z}^i. \quad (3)$$

The gauging of $\Omega\mathcal{R}$ creates Orientifold fixed planes (O-planes). The location of those planes is given by sets of points fixed under the elements of $\Omega\mathcal{R} \times \mathbb{Z}_N \times \mathbb{Z}_M$. These fixed planes carry RR-charges which must be canceled by adding D-branes to the model[1, 8]. One of these O-planes is extended along the non-compact directions and the real parts of the z^i . It is invariant under the \mathcal{R} reflection. To visualize the remaining O-planes we note that ($I = 1, 2$)

$$\Omega\mathcal{R}\Theta_I : z^i \rightarrow (\Theta_I)^{-\frac{1}{2}} \Omega\mathcal{R}(\Theta_I)^{\frac{1}{2}} : z^i \quad (4)$$

where the powers of $\pm\frac{1}{2}$ indicate that the rotations are performed with plus-minus half the angle as compared to (2). Thus, a fixed plane under $\Omega\mathcal{R}\Theta_I$ is obtained by acting with $(\Theta_I)^{-\frac{1}{2}}$ on the set of points fixed under \mathcal{R} . Therefore (starting from the O-6-plane at $z^i = \bar{z}^i$, $i = 1, 2, 3$), one obtains a set of O-6-planes intersecting at angles, whose values are given by half the order of the corresponding $\mathbb{Z}_N \times \mathbb{Z}_M$ element. Since all these O-6-planes have compact transverse directions and carry RR-charges, one expects that for consistency one needs to include a certain amount of D-6-branes canceling exactly those charges. These D-6-branes need to be parallel to the O-6-planes and hence also intersect at angles. Indeed, for \mathbb{Z}_N orbifolds this has been shown to be the case[19, 20]. In the following sections we will generalize those models to $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds. This turns out to be a straightforward modification of the discussion given in[19, 20]. A new ingredient, however, is that in some cases more complicated projective representations of the orientifold group in the open string sector are needed. This has been observed before in some $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds of type IIB models[11, 15]. In fact, for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model to be discussed in section 4 we will obtain the T-dual version of the model of[11].

In the next paragraphs we are going to review some of the technical aspects necessary for orientifold constructions. The main consistency requirement comes from RR charge conservation. Technically, it translates into the tadpole cancellation condition[8]. The RR charges describe the size of the couplings of O-planes and D-branes to RR gauge fields. The numerical

value of these couplings can be computed by extracting the RR exchange contribution to the forces acting between O-planes and D-branes[1]. A convenient way of computing these forces is to move to the open string loop channel. There, the RR charge of O-planes and D-branes can be extracted from the UV-limits of the following parts of the Klein bottle, Möbius strip and annulus diagrams[21]:

$$\begin{aligned} \text{Klein bottle:} & \quad \text{Closed string NSNS states with } \mathbf{P}\Omega\mathcal{R}(-)^F \text{ insertion} \\ \text{Möbius strip:} & \quad \text{Open string R states with } -\mathbf{P}\Omega\mathcal{R} \text{ insertion} \\ \text{Annulus:} & \quad \text{Open string NS states with } \mathbf{P}(-)^F \text{ insertion} \end{aligned} \tag{5}$$

Here, $(-)^F$ is the fermion number to be defined below. (For closed strings $(-)^F = (-)^{F_L} = (-)^{F_R}$ because of the presence of Ω in the trace.) Further, we denote by \mathbf{P} the projector on states invariant under the orbifold group $\mathbb{Z}_N \times \mathbb{Z}_M$. The requirement of tadpole cancellation determines the number of D-6-branes and part of the representation of the orientifold group on the Chan Paton indices.

Another essential consistency condition is what is called “completion of the projector in the tree channel” in[19]. Let us briefly recall their arguments. The important diagrams are drawn in figure 1. The notation is taken from[8]. Orbifold group elements are denoted by h or g . In the tree channel picture the crosscaps correspond to O-planes invariant under $\Omega\mathcal{R}h$. D-branes are assigned a letter i or j . The orbifold group element g denotes the twist sector of the closed strings propagating in the tree channel. (For further details see[8].) Consistency of the boundary conditions requires

$$(\Omega\mathcal{R}h_1)^2 = (\Omega\mathcal{R}h_2)^2 = g \tag{6}$$

in the Klein bottle diagram and

$$(\Omega\mathcal{R}h)^2 = g \tag{7}$$

in the Möbius strip. For the class of orientifolds discussed here, the lhs of (6) and (7) are the identity. Hence, in the tree channel only untwisted closed strings propagate in the Klein bottle and in the Möbius strip. Since these must be invariant under the orbifold group, the tree channel amplitude must contain the insertion of the complete projector on invariant states. The actual calculation of the diagrams depicted in figure 1 will be done in the loop channel, where worldsheet time is vertical. Transforming back to the tree channel one must recover the insertion of the complete projector mentioned above. For the Klein bottle amplitude this requirement leads to restrictions on the possible orbifold lattices. In the Möbius strip one obtains further conditions for the representation of the orientifold group on the Chan-Paton matrices.

Finally, let us introduce some notation and definitions which we mainly borrow from[20]. The action of the orientifold group on closed string degenerate ground states¹ is specified by

$$\begin{aligned} \mathcal{R} : |s_0, s_1, s_2, s_3\rangle & \rightarrow |s_0, -s_1, -s_2, -s_3\rangle \\ \Theta_1 : |s_0, s_1, s_2, s_3\rangle & \rightarrow e^{2\pi i \vec{v} \cdot \vec{s}} |s_0, s_1, s_2, s_3\rangle \\ \Theta_2 : |s_0, s_1, s_2, s_3\rangle & \rightarrow e^{2\pi i \vec{w} \cdot \vec{s}} |s_0, s_1, s_2, s_3\rangle \end{aligned} \tag{8}$$

¹All states are in light cone gauge. The first entry in the state corresponds to two non-compact directions, whereas the other three belong to the three complex compact directions.

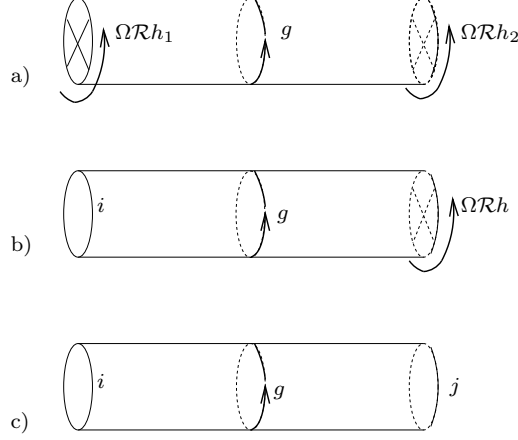


Figure 1: a) Klein bottle, b) Möbius strip, c) Cylinder

Worldsheet parity inversion Ω interchanges the left with the right moving sector. Under GSO projection states with fermion numbers $(-1)^{F_L} = (-1)^{F_R} = 1$ are kept, where

$$\begin{aligned} (-1)^{F_L} |s_0, s_1, s_2, s_3\rangle &= -e^{\pi i(s_0 - s_1 - s_2 - s_3)} |s_0, s_1, s_2, s_3\rangle \\ (-1)^{F_R} |s_0, s_1, s_2, s_3\rangle &= -e^{\pi i(s_0 + s_1 + s_2 + s_3)} |s_0, s_1, s_2, s_3\rangle \end{aligned} \quad (9)$$

In the loop channel the expressions for the Klein bottle, Möbius strip and annulus are ($c = V_4 / (8\pi\alpha')^2$ and V_4 is the regularized volume of non-compact momentum space)

$$\mathcal{K} = 4c \int_0^\infty \frac{dt}{t^3} \text{Tr}_{U+T} \left(\frac{\Omega \mathcal{R}}{2} \mathbf{P} \left(\frac{1 + (-1)^F}{2} \right) (-1)^{\mathbf{S}} e^{-2\pi t(L_0 + \bar{L}_0)} \right), \quad (10)$$

$$\mathcal{A} = c \int_0^\infty \frac{dt}{t^3} \text{Tr}_{\text{open}} \left(\frac{1}{2} \mathbf{P} \left(\frac{1 + (-1)^F}{2} \right) (-1)^{\mathbf{S}} e^{-2\pi t L_0} \right), \quad (11)$$

$$\mathcal{M} = c \int_0^\infty \frac{dt}{t^3} \text{Tr}_{\text{open}} \left(\frac{\Omega \mathcal{R}}{2} \mathbf{P} \left(\frac{1 + (-1)^F}{2} \right) (-1)^{\mathbf{S}} e^{-2\pi t L_0} \right). \quad (12)$$

Here,

$$\mathbf{P} = \left(\frac{1 + \Theta_1 + \dots + \Theta_1^{(N-1)}}{N} \right) \left(\frac{1 + \Theta_2 + \dots + \Theta_2^{(M-1)}}{M} \right) \quad (13)$$

is the projector on states invariant under the orbifold group $\mathbb{Z}_N \times \mathbb{Z}_M$. \mathbf{S} denotes the space time fermion number. In order to compute the contribution due to RR exchange in the tree channel one needs to compute the parts of expressions (10), (11) and (12) which are given in (5). (The space time fermion number insertion has been taken care of by the minus sign in the second line in (5).) The transformation back to the tree channel is performed by replacing $t = \frac{1}{4l}$ for the Klein bottle, $t = \frac{1}{8l}$ for the Möbius strip and $t = \frac{1}{2l}$ for the annulus[8]. Finally RR-charge conservation is imposed by demanding the infrared ($l \rightarrow \infty$) limit of the tree channel expression to be finite.

3 The $\mathbb{Z}_4 \times \mathbb{Z}_2$ $\Omega\mathcal{R}$ -Orientifold

In this section we discuss the $\mathbb{Z}_4 \times \mathbb{Z}_2$ model in great detail because in this case all possible subtleties show up; therefore the other models can be treated briefly in the following sections.

The lattice described by the shifts $\vec{v} = (1/4, -1/4, 0)$ for the \mathbb{Z}_4 -factor and $\vec{w} = (0, 1/2, -1/2)$ for the \mathbb{Z}_2 -factor of the orbifold-group is essentially an $SU(2)^6$ -lattice, i.e. a product of three tori in the notation of complex compact coordinates. The two crystallographically allowed orientations **A** and **B** of one of the three tori with respect to the reflection \mathcal{R} are shown in figure 2. As will be explained in the following, the only perturbatively consistent models are given by the choices **ABA** and **ABB** for the compact directions².

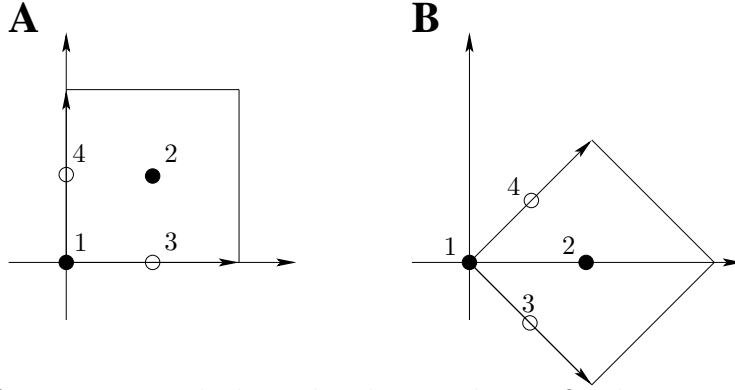


Figure 2: Lattices for $\mathbb{Z}_4 \times \mathbb{Z}_2$. Black circles denote the \mathbb{Z}_4 fixed points and white circles the additional \mathbb{Z}_2 fixed points.

3.1 The Klein bottle amplitude

We begin with evaluating the general expression (10) for the Klein bottle 1-loop amplitude of the $\mathbb{Z}_4 \times \mathbb{Z}_2$ model by considering the compact momenta. The Kaluza-Klein (KK) and winding (W) states are generally given by

$$P = \frac{\sqrt{2}}{r} (m_1 \vec{e}_1^* + m_2 \vec{e}_2^*) , \quad (14)$$

$$W = \frac{r}{\sqrt{2}\alpha'} (n_1 \vec{e}_1 + n_2 \vec{e}_2) , \quad (15)$$

where m_i and n_i are integers, \vec{e}_i are the basis vectors of the corresponding torus with radius r and \vec{e}_i^* are the basis vectors of the dual torus ($i = 1, 2$). The $SU(2)^2$ lattices in figure 2 are spanned by

$$\vec{e}_1^{\mathbf{A}} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} , \quad \vec{e}_2^{\mathbf{A}} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} , \quad (16)$$

$$\vec{e}_1^{\mathbf{B}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} , \quad \vec{e}_2^{\mathbf{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad (17)$$

²The choices **BAA** and **BAB** are equivalent to these models.

with the corresponding dual basis. For the Kaluza-Klein and winding states invariant under $\Omega\mathcal{R}$ one gets

$$P^{\mathbf{A}} = \frac{m}{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad W^{\mathbf{A}} = \frac{nr}{\alpha'} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (18)$$

$$P^{\mathbf{B}} = \frac{\sqrt{2}m}{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad W^{\mathbf{B}} = \frac{\sqrt{2}nr}{\alpha'} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (19)$$

where m, n are integers. As a consequence of relation (4) the states (18) and (19) are invariant under the insertions $\Omega\mathcal{R}\Theta_1^{k_1}\Theta_2^{k_2}$ for $k_1 = 0, 2$ and $k_2 = \text{arbitrary}$; when \mathbf{A} and \mathbf{B} are exchanged, these states are invariant under insertions with $k_1 = 1, 3$ and $k_2 = \text{arbitrary}$. From

$$p_{L,R} = P \pm W \quad (20)$$

for the closed string, it follows that the lattice contribution to the 1-loop amplitude for the Klein bottle is $\mathcal{L}[1, 1]$ for \mathbf{A} -states and $\mathcal{L}[2, 2]$ for \mathbf{B} -states, where the notation is taken from [19, 20] and explained in appendix A. In general, lattice contributions only appear for untwisted tori.

The calculation of the oscillator contributions to (10) simplifies if one takes into account that the RR-exchange in the tree level is given by the trace over the NSNS-sector with the insertion $(-1)^F$ in the 1-loop channel. Furthermore, the elements of the orbifold group $\mathbb{Z}_4 \times \mathbb{Z}_2$ act as the unit operator on the oscillator states which contribute to the trace because $\Omega\mathcal{R}$ -invariance leads to a cancellation of the phases given in equation (8) between left- and right-movers. This means that the oscillator contributions are equal for any insertion from the orbifold group. Although the numerical results may be zero from case to case, all the twisted sectors formally show up in the amplitude, because $\Omega\mathcal{R}$ does not exchange them. The last ingredients we need are the multiplicities $\chi_{\mathcal{K}}^{(n_1, k_1)(n_2, k_2)}$ of the $\Theta_1^{n_1}\Theta_2^{n_2}$ -twisted fixed points which are invariant under the insertion $\Omega\mathcal{R}\Theta_1^{k_1}\Theta_2^{k_2}$. Consider e.g. the second torus T_2 twisted by Θ_2 . In the \mathbf{A} -lattice, two of the four fixed points are interchanged under $\Omega\mathcal{R}\Theta_1^{k_1}\Theta_2^{k_2}$ when $k_1 = 0, 2$ and $k_2 = \text{arbitrary}$. In the \mathbf{B} -lattice all four fixed points are invariant under these insertions, such that $\chi_{\mathcal{K}}^{(0,0)(1, k_2)} = \chi_{\mathcal{K}}^{(0,2)(1, k_2)} = 2(4)$ for an $\mathbf{A}(\mathbf{B})$ -type T_2 . The resulting multiplicities are shown in table 1. Considering all this, we can evaluate the Klein bottle 1-loop amplitude (10). The notation is similar to [19, 20] and defined in appendix A. We use the fact that for the oscillator contributions $\mathcal{K}^{(n_1, k_1)(n_2, k_2)} = \mathcal{K}^{(n_1, 0)(n_2, 0)}$ is valid for all n_i, k_i ($i = 1, 2$) as explained above and simplify the notation by defining $\mathcal{K}^{(n_1, n_2)} \equiv \mathcal{K}^{(n_1, 0)(n_2, 0)}$. For the \mathbf{ABA} -lattice we get

$$\begin{aligned} \mathcal{K} = c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \times \\ \left(\mathcal{L}[1, 1]^2 \mathcal{L}[2, 2] \mathcal{K}^{(0,0)} + 4\mathcal{L}[1, 1] \mathcal{K}^{(1,0)} + 8\mathcal{L}[1, 1] \mathcal{K}^{(2,0)} + 4\mathcal{L}[1, 1] \mathcal{K}^{(3,0)} \right. \\ \left. + 8\mathcal{L}[1, 1] \mathcal{K}^{(0,1)} + 16\mathcal{K}^{(1,1)} + 16\mathcal{L}[2, 2] \mathcal{K}^{(2,1)} + 16\mathcal{K}^{(3,1)} \right). \end{aligned} \quad (21)$$

For the \mathbf{ABB} -lattice, one $\mathcal{L}[1, 1]$ in every term in the second line of equation (21) has to be exchanged for $\mathcal{L}[2, 2]$ and the prefactors in the third line have to be divided by two.

$\chi\kappa$	ABA	ABB
$(0, k_1)(0, k_2)$	1	1
$(2n_1 + 1, k_1)(0, k_2)$	4	4
$(2, k_1)(0, k_2)$	8	8
$(0, 2k_1)(1, k_2)$	8	4
$(0, 2k_1 + 1)(1, k_2)$	16	8
$(2n_1 + 1, k_1)(1, k_2)$	16	8
$(2, 2k_1)(1, k_2)$	16	8
$(2, 2k_1 + 1)(1, k_2)$	8	4

Table 1: Multiplicities of the fixed points for $\mathbb{Z}_4 \times \mathbb{Z}_2$.

The modular transformation to the tree-channel $t = \frac{1}{4l}$ yields (see appendix A)

$$\begin{aligned}
\tilde{\mathcal{K}} = 32c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty dl \times \\
\left(\tilde{\mathcal{L}}[4, 4]^2 \tilde{\mathcal{L}}[2, 2] \tilde{\mathcal{K}}^{(0,0)} - 2\tilde{\mathcal{L}}[4, 4] \tilde{\mathcal{K}}^{(1,0)} - 4\tilde{\mathcal{L}}[4, 4] \tilde{\mathcal{K}}^{(2,0)} - 2\tilde{\mathcal{L}}[4, 4] \tilde{\mathcal{K}}^{(3,0)} \right. \\
\left. - 4\tilde{\mathcal{L}}[4, 4] \tilde{\mathcal{K}}^{(0,1)} + 4\tilde{\mathcal{K}}^{(1,1)} - 4\tilde{\mathcal{L}}[2, 2] \tilde{\mathcal{K}}^{(2,1)} - 4\tilde{\mathcal{K}}^{(3,1)} \right). \quad (22)
\end{aligned}$$

For the **ABB**-lattice, one $\tilde{\mathcal{L}}[4, 4]$ in every term in the second line of equation (22) again has to be exchanged for $\tilde{\mathcal{L}}[2, 2]$ and equation (22) has to be multiplied by an overall factor of $1/2$. We realize that the complete projector in the sense of [19] and section 2 shows up, because all possible insertions of the orbifold group appear, only untwisted sectors contribute and the prefactors are given by

$$\prod_{k_1 v_i + k_2 w_i \neq 0; i=1,2,3} \left(-2 \sin(\pi k_1 v_i + \pi k_2 w_i) \right), \quad (23)$$

as expected. At this point we can clarify, why only the models **ABA** and **ABB** (and the equivalent models **BAA** and **BAB**) are perturbatively consistent. The lattice contribution of e.g. **AAA** is changed to **BBA** by the insertion of Θ_1 and there is no way to get the complete projector. From the same argument it follows that only the orbifold groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_6 \times \mathbb{Z}_3$ can lead to perturbatively consistent solutions.

3.2 The annulus amplitude

To cancel the tadpoles which arise in the Klein bottle amplitude we need to introduce D-branes. As was found in [19, 20] and explained in section 2 for the orientifold models under consideration we have to introduce D-6-branes rotated by half the angles which are given by the elements of

the orbifold group. For $\mathbb{Z}_4 \times \mathbb{Z}_2$ this leads to a configuration of eight different D-6-branes whose locations in the three compact tori are shown in figure 3. For simplicity we restrict ourselves to the case where the D-branes are located at the fixed points. Writing down the mode expansions e.g. for an open string stretching from brane $(0,0)$ to brane $(1,0)$ (where the brane $(1,0)$ is rotated by $\Theta_1^{-1/2}$ with respect to the brane $(0,0)$) one realizes that the modings are the same as for the closed string twisted by Θ_1^{-1} . Therefore it is convenient to call these kinds of open strings “twisted sectors” [19], as we will do in the following. Using these conventions, an open string stretching from brane (i_1, i_2) to brane $(i_1 - n_1, i_2 - n_2)$ belongs to the $\Theta_1^{n_1} \Theta_2^{n_2}$ -twisted sector.

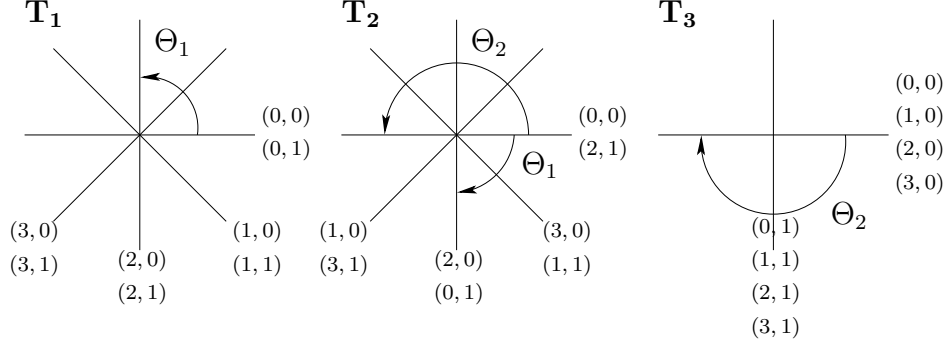


Figure 3: Arrangement of branes and action of the orbifold group for $\mathbb{Z}_4 \times \mathbb{Z}_2$. The branes are labelled by $(i_1, i_2) = (0, 0), \dots, (3, 1) \bmod (4, 2)$. This is convenient in the sense that the brane (i_1, i_2) is obtained by rotating the compact real axes with $\Theta_1^{-i_1/2} \Theta_2^{-i_2/2}$, see also section 2.

For the open string the compact momenta p_{open} are given by the distance of parallel D-6-branes in the corresponding directions. Therefore we consider the location of the branes in the fundamental cells of the lattices **A** and **B**, see figure 4. Starting from brane $(0,0)$, we

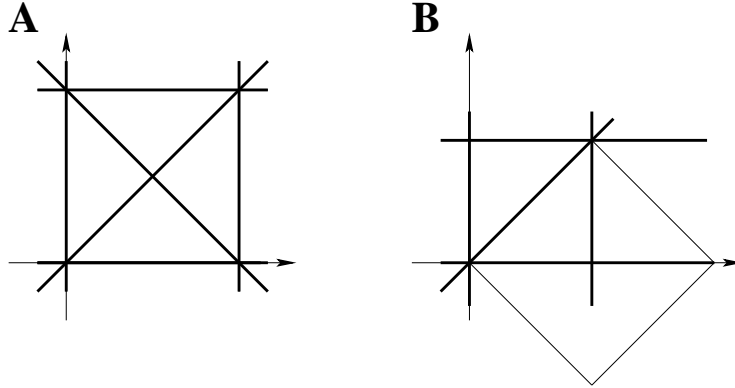


Figure 4: Location of branes in the fundamental cells for $\mathbb{Z}_4 \times \mathbb{Z}_2$

get $p_{open}^{\mathbf{A}} = P^{\mathbf{A}}$ for the directions 4,6,8 and $p_{open}^{\mathbf{A}} = W^{\mathbf{A}}$ for the directions 5,7,9 as well as $p_{open}^{\mathbf{B}} = \frac{1}{2}P^{\mathbf{B}}$ (4,6,8) and $p_{open}^{\mathbf{B}} = \frac{1}{2}W^{\mathbf{B}}$ (5,7,9), see equations (18) and (19). It follows, that the lattice contribution to the 1-loop amplitude for the annulus is given by $\mathcal{L}[2, 2]$ for an **A**-torus and by $\mathcal{L}[1, 1]$ for a **B**-torus. The compact momenta are nonzero again only for untwisted tori, i.e. tori where the twist acts trivially. But in addition, the D-branes have to be invariant under

insertions of the elements of the orbifold group, therefore only the insertions $\mathbb{1}$, Θ_1^2 , Θ_2 and $\Theta_1^2\Theta_2$ yield non-vanishing compact momenta on untwisted tori.

Calculating the oscillator contributions to (11) we again focus on the RR-exchange in the tree channel which is given by the trace over the NS-sector with $(-1)^F$ insertion in the 1-loop amplitude. All twisted sectors appear, each in combination with the insertions $\mathbb{1}$, Θ_1^2 , Θ_2 and $\Theta_1^2\Theta_2$ which leave the D-branes invariant, as stated above. In contrast to the Klein bottle amplitude the phases arising from the insertions are not cancelled. Moreover, the representation matrices of the orbifold group have to be taken into account. These matrices are unitary $M \times M$ matrices, where M is the number of arrangements that will be fixed by the tadpole cancellation conditions. In the amplitude the matrices appear as

$$\text{tr}\gamma_{k_1 k_2}^{(i_1-n_1, i_2-n_2)} \text{tr}\gamma_{k_1 k_2}^{(i_1, i_2)-1}, \quad (24)$$

where (i_1, i_2) labels the eight different branes and n_1 and n_2 indicate the $\Theta_1^{n_1}\Theta_2^{n_2}$ -twisted sector, as explained above. $\gamma_{k_1 k_2}^{(i_1, i_2)}$ is an abbreviation for $\gamma_{\Theta_1^{k_1}\Theta_2^{k_2}}^{(i_1, i_2)}$. This leads to a factor of $8M^2$ for the untwisted and twisted sectors without insertions (“without” means insertion of $\mathbb{1}$ in the case of the annulus), where the 8 arises from the number of branes in the arrangement (see figure 3). In fact, all the calculations can be done starting with brane $(0, 0)$ and inserting factors of 8 appropriately, because the other branes lead to the same amplitudes.

The terms with insertions lead to twisted sector tadpoles in the tree-channel, which cannot be cancelled by the other diagrams. This yields the twisted sector tadpole cancellation conditions

$$\text{tr}\gamma_{20}^{(i_1, i_2)} = \text{tr}\gamma_{01}^{(i_1, i_2)} = \text{tr}\gamma_{21}^{(i_1, i_2)} = 0 \quad (25)$$

for all (i_1, i_2) , similar to [8].

The analogue to the multiplicities of fixed points in the closed string are the intersection numbers of the D-branes in the open string case. The intersection numbers are given by the number of times that the branes intersect within the fundamental cell and can be read off easily from figure 4. Starting with brane $(0, 0)$, a \mathbb{Z}_2 -twisted **B**-type torus contributes a factor of two, whereas a \mathbb{Z}_2 -twisted **A**-type torus and \mathbb{Z}_4 -twisted tori of both types contribute a factor of one³. Again, only the points invariant under insertions contribute, thus in the case of the annulus only sectors without insertions appear in the amplitude and it is sufficient to consider the multiplicities $\chi_{\mathcal{A}}^{(n_1, n_2)} \equiv \chi_{\mathcal{A}}^{(n_1, 0)(n_2, 0)}$ which are given in table 2. Now we have all the ingredients to write down the annulus 1-loop amplitude for the **ABA**-lattice

$$\begin{aligned} \mathcal{A} = M^2 \frac{c}{4} (1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \times \\ \left(\mathcal{L}[1, 1] \mathcal{L}[2, 2]^2 \mathcal{A}^{(0, 0)} + \mathcal{L}[2, 2] \mathcal{A}^{(1, 0)} + 2\mathcal{L}[2, 2] \mathcal{A}^{(2, 0)} + \mathcal{L}[2, 2] \mathcal{A}^{(3, 0)} \right. \\ \left. + 2\mathcal{L}[2, 2] \mathcal{A}^{(0, 1)} + \mathcal{A}^{(1, 1)} + \mathcal{L}[1, 1] \mathcal{A}^{(2, 1)} + \mathcal{A}^{(3, 1)} \right), \end{aligned} \quad (26)$$

where we used the simplified notation $\mathcal{A}^{(n_1, n_2)} \equiv \mathcal{A}^{(n_1, 0)(n_2, 0)}$ (see appendix A). All the terms with insertions vanish due to the twisted tadpole cancellation condition (25) and therefore do

³Remember that we are discussing strings starting on brane $(0, 0)$.

$\chi_{\mathcal{A}}$	ABA	ABB
(0, 0)	1	1
(1, 0)	1	1
(2, 0)	2	2
(3, 0)	1	1
(0, 1)	2	4
(1, 1)	1	2
(2, 1)	1	2
(3, 1)	1	2

Table 2: Intersection numbers in the annulus for $\mathbb{Z}_4 \times \mathbb{Z}_2$.

not appear in equation (26). For the **ABB**-lattice one $\mathcal{L}[2, 2]$ in every term in the second line of equation (26) has to be exchanged for $\mathcal{L}[1, 1]$ and the prefactors in the third line have to be multiplied by two.

Performing the modular transformation $t = \frac{1}{2l}$ leads to

$$\begin{aligned}
\tilde{\mathcal{A}} = & \frac{c}{8} M^2 (1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty dl \times \\
& \left(\tilde{\mathcal{L}}[1, 1]^2 \tilde{\mathcal{L}}[2, 2] \tilde{\mathcal{A}}^{(0,0)} - 2\tilde{\mathcal{L}}[1, 1] \tilde{\mathcal{A}}^{(1,0)} - 4\tilde{\mathcal{L}}[1, 1] \tilde{\mathcal{A}}^{(2,0)} - 2\tilde{\mathcal{L}}[1, 1] \tilde{\mathcal{A}}^{(3,0)} \right. \\
& \left. - 4\tilde{\mathcal{L}}[1, 1] \tilde{\mathcal{A}}^{(0,1)} + 4\tilde{\mathcal{A}}^{(1,1)} - 4\tilde{\mathcal{L}}[2, 2] \tilde{\mathcal{A}}^{(2,1)} - 4\tilde{\mathcal{A}}^{(3,1)} \right). \tag{27}
\end{aligned}$$

For the **ABB**-lattice, one $\tilde{\mathcal{L}}[1, 1]$ in every term in the second line of equation (27) has to be exchanged for $\tilde{\mathcal{L}}[2, 2]$ and the whole amplitude has to be multiplied by two. Again, the complete projector shows up.

3.3 The Möbius strip amplitude

In the case of the Möbius strip, the compact momenta for a **B**-torus have to be doubled in the 5,7,9 directions because of the $\Omega\mathcal{R}$ -projection, therefore one gets $\mathcal{L}[2, 2]$ for an **A**-torus and $\mathcal{L}[1, 4]$ for a **B**-torus. Again, the lattice contributions only appear for untwisted tori and for insertions which leave the D-6-branes invariant.

Which strings contribute to the Möbius strip one-loop amplitude? Let us denote a $\Theta_1^{n_1} \Theta_2^{n_2}$ -twisted string by $[(i_1, i_2)(i_1 - n_1, i_2 - n_2)]$ as explained in section 3.2 and consider the action of

the insertion $\Omega\mathcal{R}\Theta_1^{k_1}\Theta_2^{k_2}$ thereupon:

$$\begin{aligned}
& [(i_1, i_2)(i_1 - n_1, i_2 - n_2)] \\
& \xrightarrow{\Theta_1^{k_1}\Theta_2^{k_2}} [(i_1 + 2k_1, i_2 + 2k_2)(i_1 - n_1 + 2k_1, i_2 - n_2 + 2k_2)] \\
& \xrightarrow{\mathcal{R}} [(-i_1 - 2k_1, -i_2 - 2k_2)(-i_1 + n_1 - 2k_1, -i_2 + n_2 - 2k_2)] \\
& \xrightarrow{\Omega} [(-i_1 + n_1 - 2k_1, -i_2 + n_2 - 2k_2)(-i_1 - 2k_1, -i_2 - 2k_2)] .
\end{aligned} \tag{28}$$

Since $n_2, k_2 = 0, 1 \pmod{2}$, the condition $i_2 = -i_2 + n_2 - 2k_2 \pmod{2}$ is equivalent to $2i_2 = n_2 \pmod{2}$, thus only \mathbb{Z}_2 -untwisted sectors (i.e. $n_2 = 0$) with $k_2 = \text{arbitrary}$ contribute to the amplitude. The condition $i_1 = -i_1 + n_1 - 2k_1 \pmod{4}$ implies e.g. for the brane with $i_1 = 0$ that the sectors $n_1 = 0$ with $k_1 = 0, 2$ and $n_1 = 2$ with $k_1 = 1, 3$ contribute to the amplitude. To summarize, for $(i_1, i_2) = (0, 0)$ the sectors $(n_1, k_1)(n_2, k_2) = (0, 0)(0, 0), (0, 0)(0, 1), (0, 2)(0, 0), (0, 2)(0, 1), (2, 1)(0, 0), (2, 1)(0, 1), (2, 3)(0, 0)$ and $(2, 3)(0, 1)$ contribute. The other branes of the arrangement in figure 3 get contributions from different sectors, but the resulting amplitude is the same, therefore we can restrict the calculation to the case $(i_1, i_2) = (0, 0)$ and insert factors of 8 appropriately, again.

Now the representation matrices of the orientifold group have to be taken into account. For the brane (i_1, i_2) in the sector $(n_1, k_1)(n_2, k_2)$ they appear as

$$\text{tr} \left[\gamma_{\Omega\mathcal{R}k_1k_2}^{(i_1-n_1, i_2-n_2)-T} \gamma_{\Omega\mathcal{R}k_1k_2}^{(i_1, i_2)} \right] . \tag{29}$$

Since only untwisted and Θ_1^2 -twisted sectors appear, we abbreviate $a_{k_1k_2}^{(n_1)} \equiv \text{tr} \left[\gamma_{\Omega\mathcal{R}k_1k_2}^{(n_1, 0)-T} \gamma_{\Omega\mathcal{R}k_1k_2}^{(0, 0)} \right]$. The multiplicities $\chi_{\mathcal{M}}$ can be obtained in the same way as in the case of the annulus, because all intersection points are invariant under $\Omega\mathcal{R}$. We get $\chi_{\mathcal{M}} = 2$ in the Θ_1^2 -twisted sectors and $\chi_{\mathcal{M}} = 1$ in the other sectors.

This leads to the Möbius strip 1-loop amplitude for the **ABA**-lattice

$$\begin{aligned}
\mathcal{M} = & -\frac{c}{4} (1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \times \\
& \left(a_{00}^{(0)} \mathcal{L}[1, 4] \mathcal{L}[2, 2]^2 \mathcal{M}^{(0,0)(0,0)} + a_{01}^{(0)} \mathcal{L}[2, 2] \mathcal{M}^{(0,0)(0,1)} \right. \\
& + 2a_{10}^{(2)} \mathcal{L}[2, 2] \mathcal{M}^{(2,1)(0,0)} + 2a_{11}^{(2)} \mathcal{M}^{(2,1)(0,1)} + a_{20}^{(0)} \mathcal{L}[2, 2] \mathcal{M}^{(0,2)(0,0)} \\
& \left. + a_{21}^{(0)} \mathcal{L}[1, 4] \mathcal{M}^{(0,2)(0,1)} + 2a_{30}^{(2)} \mathcal{L}[2, 2] \mathcal{M}^{(2,3)(0,0)} + 2a_{31}^{(2)} \mathcal{M}^{(2,3)(0,1)} \right) .
\end{aligned} \tag{30}$$

For the **ABB**-lattice, one $\mathcal{L}[2, 2]$ in each term of equation (30) with $a_{k_1k_2}^{(n_1)} = a_{k_10}^{(n_1)}$ has to be exchanged for $\mathcal{L}[1, 4]$.

Performing the transformation to the tree-channel $t = \frac{1}{8l}$ (see appendix A) yields

$$\begin{aligned}
\tilde{\mathcal{M}} = & -4c (1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty dl \times \\
& \left(a_{00}^{(0)} \tilde{\mathcal{L}}[8, 2] \tilde{\mathcal{L}}[4, 4]^2 \tilde{\mathcal{M}}^{(0,0)} - 2a_{30}^{(2)} \tilde{\mathcal{L}}[4, 4] \tilde{\mathcal{M}}^{(1,0)} \right. \\
& + 4a_{20}^{(0)} \tilde{\mathcal{L}}[4, 4] \tilde{\mathcal{M}}^{(2,0)} - 2a_{10}^{(2)} \tilde{\mathcal{L}}[4, 4] \tilde{\mathcal{M}}^{(3,0)} + 4a_{01}^{(0)} \tilde{\mathcal{L}}[4, 4] \tilde{\mathcal{M}}^{(0,1)} \\
& \left. + 4a_{31}^{(2)} \tilde{\mathcal{M}}^{(1,1)} + 4a_{21}^{(0)} \tilde{\mathcal{L}}[8, 2] \tilde{\mathcal{M}}^{(2,1)} + 4a_{11}^{(2)} \tilde{\mathcal{M}}^{(3,1)} \right) .
\end{aligned} \tag{31}$$

For the **ABB**-lattice, one $\mathcal{L}[4, 4]$ in each term of equation (31) with $a_{k_1 k_2}^{(n_1)} = a_{k_1 0}^{(n_1)}$ has to be exchanged for $\mathcal{L}[8, 2]$.

To obtain the complete projector and to cancel the untwisted tadpoles from the other diagrams, the γ -matrices have to fulfill the conditions

$$a_{00}^{(0)} = a_{10}^{(2)} = -a_{20}^{(0)} = a_{30}^{(2)} = -a_{01}^{(0)} = -a_{11}^{(2)} = -a_{21}^{(0)} = a_{31}^{(2)} = M \quad (32)$$

and the untwisted tadpole cancellation condition reads

$$\begin{aligned} \mathbf{ABA}: \quad & [M - 16]^2 = 0, \\ \mathbf{ABB}: \quad & [M - 8]^2 = 0, \end{aligned} \quad (33)$$

which fixes the number of arrangements shown in figure 3 to be 16(8) for the **ABA** (**ABB**) lattice, respectively. The conditions (32) are valid for the brane $(i_1, i_2) = (0, 0)$. For some other brane (i_1, i_2) one has to replace the insertions $\Theta_1^{k_1} \Theta_2^{k_2}$ in (32) by $\Theta_1^{k_1+i_1} \Theta_2^{k_2+i_2}$.

3.4 The closed string spectrum

The massless spectrum is found by symmetrizing the massless states which satisfy the GSO-projection conditions (9) with respect to $\Omega\mathcal{R}$, Θ_1 and Θ_2 . Ω exchanges left- and right-movers and is defined following the convention of [8]

$$\Omega\alpha_r\Omega^{-1} = \tilde{\alpha}_r, \quad \Omega\psi_r\Omega^{-1} = \tilde{\psi}_r, \quad \Omega\tilde{\psi}_r\Omega^{-1} = -\psi_r \quad (34)$$

for integer and half-integer r . For the action of \mathcal{R} , Θ_1 and Θ_2 see section 2. In the following we denote the NSNS vacuum by $|0\rangle$ and e.g. the R state $|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle_L$ by $|++++\rangle_L$. The states are given up to normalization. In the untwisted sector we find the massless states

NSNS:	$(\psi^\mu \tilde{\psi}^\nu + \psi^\nu \tilde{\psi}^\mu) 0\rangle$ $\psi^i \tilde{\psi}^{\bar{i}} 0\rangle, \psi^{\bar{i}} \tilde{\psi}^i 0\rangle$ $(\psi^3 \tilde{\psi}^3 + \psi^{\bar{3}} \tilde{\psi}^{\bar{3}}) 0\rangle$	graviton + dilaton (m.i.)	
		$(i = 1, 2, 3; \bar{i} = \bar{1}, \bar{2}, \bar{3})$ 6 scalars (m.i.)	
		1 scalar	
RR:	$ ++++\rangle_L ----\rangle_R - ----\rangle_L ++++\rangle_R$ $ +-+-\rangle_L +-+-\rangle_R - +-+ \rangle_L -+-\rangle_R$	axion (m.i.)	
		1 scalar	

where (m.i.) stands for “model independent” states, i.e. states which are present independent of the orbifold group. To summarize, the untwisted massless closed string spectrum contains the $\mathcal{N} = 1$ supergravity multiplet in $D = 4$ and 4C, where C denotes the chiral multiplet.

In the $\Theta_1^{n_1} \Theta_2^{n_2}$ -twisted sectors the masses are given by

$$\frac{\alpha'}{4} m_{L,R}^2 = N_{L,R} + \frac{1}{2} q_{L,R}^2 + E_{vac} - \frac{1}{2}, \quad (35)$$

with

$$q_{L,R} = \begin{cases} (0, \pm(n_1 \vec{v} + n_2 \vec{w})) & (\text{NS}) \\ (\frac{1}{2}, \frac{1}{2} \pm (n_1 \vec{v} + n_2 \vec{w})) & (\text{R}) \end{cases} \quad (36)$$

and

$$E_{vac} = \frac{1}{2} \sum_{i=1,2,3} |n_1 v_i + n_2 w_i| (1 - |n_1 v_i + n_2 w_i|) , \quad (37)$$

where one has to take care of $0 \leq |n_1 v_i + n_2 w_i| < 1$. We discuss the Θ_1^2 -twisted sector explicitly, the other sectors are obtained in a similar manner.

In the Θ_1^2 -twisted NSNS sector, the massless states which fulfill the GSO-projection (9) are found to be

$$\begin{aligned} |0, \tfrac{1}{2}, \tfrac{1}{2}, 0\rangle_L &\equiv |1\rangle_{NS} , & |0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0\rangle_L &\equiv |2\rangle_{NS} , \\ |0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0\rangle_R &\equiv |\tilde{1}\rangle_{NS} , & |0, \tfrac{1}{2}, \tfrac{1}{2}, 0\rangle_R &\equiv |\tilde{2}\rangle_{NS} , \end{aligned}$$

such that we get the two massless ground states $|1\tilde{1}\rangle_{NSNS}$ and $|2\tilde{2}\rangle_{NSNS}$. The other two possible combinations are not invariant under Θ_2 . Furthermore we have to consider the action of $\Omega\mathcal{R}$, Θ_1 and Θ_2 on the fixed points. Since the torus T_3 is untwisted, the discussion is valid for both the **ABA** and the **ABB** lattice. For the fixed point structure see figure 2. In the tori T_1 and T_2 , the fixed points 3 and 4 are interchanged by Θ_1 . In addition, the fixed points 3 and 4 in the torus T_2 are interchanged by $\Omega\mathcal{R}$. Under the action of Θ_2 all the fixed points are invariant. In the following e.g. $\{13\}$ denotes the fixed point built from fixed points 1 of T_1 and 3 of T_2 . The fixed points $\{11\}$, $\{12\}$, $\{21\}$ and $\{22\}$ are invariant under $\Omega\mathcal{R}$ and Θ_1 . The fixed points $\{31\}$, $\{41\}$ as well as $\{32\}$, $\{42\}$ form pairs under Θ_1 . The fixed points $\{13\}$, $\{14\}$ as well as $\{23\}$, $\{24\}$ form pairs under Θ_1 and $\Omega\mathcal{R}$. The remaining fixed points $\{33\}$, $\{34\}$, $\{43\}$ and $\{44\}$ form a quartet under Θ_1 and $\Omega\mathcal{R}$. Symmetrization in the NSNS sector leads to two scalars for each fixed point, each pair and the quartet, i.e. 18 scalars altogether.

The discussion of the Θ_1^2 -twisted RR sector is similar, but here the action of $\Omega\mathcal{R}$ gives an additional minus sign. Therefore we have to antisymmetrize between left and right movers, such that we get no states from the fixed points and pairs mentioned above. The quartet contributes one vector (V).

Adding the superpartners from the NSR sector, we find $9C + 1V$ in the Θ_1^2 -twisted sector. The remaining twisted sectors can be treated in a similar fashion and we obtain the massless twisted closed string spectrum

$$\begin{aligned} \text{ABA:} & \quad 57C + 1V , \\ \text{ABB:} & \quad 47C + 11V . \end{aligned} \quad (38)$$

3.5 The open string spectrum

In order to determine the open string spectrum we have to count the degrees of freedom of the Chan-Paton factors for the massless states. Therefore, we have to find a representation of the orientifold group which satisfies the tadpole cancellation conditions (25) and (32). In general, we have to consider the action of the orientifold group on massless states of the form $|\psi, ij\rangle \lambda_{ji}^{(a,b)}$, where ψ represents the vacuum together with some combination of oscillators, $i, j = 1, \dots, M$ (M is the number of arrangements of branes) and $\lambda^{(a,b)}$ is the Chan-Paton matrix for a string

starting on brane a and ending on brane b , with $a, b = 1, \dots, 8$, i.e.⁴

$$(\Omega\mathcal{R}\Theta_1^{k_1}\Theta_2^{k_2}) : |\psi, ij\rangle \lambda_{ji}^{(a,b)} \longrightarrow |\Omega\mathcal{R}\Theta_1^{k_1}\Theta_2^{k_2} \cdot \psi, ij\rangle \left(\gamma_{\Omega\mathcal{R}k_1k_2}^{(b)} \lambda_{\Omega\mathcal{R}k_1k_2}^{(a)-1} \right)_{ji}^T. \quad (39)$$

To check whether the twisted tadpole cancellation conditions (25) are satisfied, we need the representation matrices of the orbifold group which can be obtained via e.g.

$$(\Omega\mathcal{R}\Theta_1^{k_1-1}\Theta_2^{k_2})(\Omega\mathcal{R}\Theta_1^{k_1}\Theta_2^{k_2}) : |\psi, ij\rangle \lambda_{ji}^{(a,b)} = \Theta_1 : |\psi, ij\rangle \lambda_{ji}^{(a,b)} \quad (40)$$

which implies $\gamma_1^{(a)} \simeq \gamma_{\Omega\mathcal{R}k_1-1,k_2}^{(b)-T} \gamma_{\Omega\mathcal{R}k_1k_2}^{(a)}$, where “ \simeq ” means equal up to an irrelevant phase. Taking into account all these constraints, we find the γ -matrices listed in appendix C. They form a projective representation of the orientifold group⁵, where the $\mathbb{Z}_2 \times \mathbb{Z}_2$ substructure is similar to the model discussed in [11], which in turn is T-dual to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\Omega\mathcal{R}$ -orientifold discussed in section 4.

In the untwisted NS sector (i.e. strings which start and end on the same brane) the massless states are given by $\psi_{-1/2}^m |0, ij\rangle \lambda_{ji}^{(a,a)}$ ($m = 0, \dots, 9$). The string $(1, 1)$ is invariant under $\Omega\mathcal{R}$, $\Omega\mathcal{R}\Theta_1^2$, $\Omega\mathcal{R}\Theta_2$, and $\Omega\mathcal{R}\Theta_1^2\Theta_2$, where the last symmetry is not independent of the first three. Applying equation (39), we get the following constraints on the Chan-Paton matrix $\lambda^{(1,1)}$ in the noncompact directions $\mu = 0, \dots, 3$

$$\lambda^{(1,1)} = - \left(\gamma_{\Omega\mathcal{R}00}^{(1)} \lambda^{(1,1)} \gamma_{\Omega\mathcal{R}00}^{(1)-1} \right)^T = - \left(\gamma_{\Omega\mathcal{R}20}^{(1)} \lambda^{(1,1)} \gamma_{\Omega\mathcal{R}20}^{(1)-1} \right)^T = - \left(\gamma_{\Omega\mathcal{R}01}^{(1)} \lambda^{(1,1)} \gamma_{\Omega\mathcal{R}01}^{(1)-1} \right)^T, \quad (41)$$

where the minus signs arise from the action of $\Omega\mathcal{R}$ on the massless state. Using the γ -matrices listed in appendix C this leads to

$$\lambda^{(1,1)} = -\lambda^{(1,1)T} = M_1 \lambda^{(1,1)T} M_1 = M_2 \lambda^{(1,1)T} M_2. \quad (42)$$

Counting the remaining degrees of freedom of $\lambda^{(1,1)}$ we find that $\psi_{-1/2}^\mu |0, ij\rangle \lambda_{ji}^{(1,1)}$ is a vector in the adjoint representation of the gauge group $Sp(\frac{M}{4})$. In a similar manner the compact directions $\psi_{-1/2}^{i,\bar{i}} |0, kl\rangle \lambda_{lk}^{(1,1)}$, ($i = 1, 2, 3; \bar{i} = \bar{1}, \bar{2}, \bar{3}$) yield 3C in the antisymmetric representation of $Sp(\frac{M}{4})$.

For the $(2, 2)$ -string, the symmetries are $\Omega\mathcal{R}\Theta_1$, $\Omega\mathcal{R}\Theta_1^3$ and $\Omega\mathcal{R}\Theta_1\Theta_2$. Inserting the corresponding γ -matrices also leads to equation (42) for $\lambda^{(2,2)}$, i.e. the same result as for the $(1, 1)$ -string, but now the gauge group $Sp(\frac{M}{4})$ constitutes a second factor of a product gauge group, since the strings are not mapped onto each other by any symmetry of the theory.

The invariances of the $(3, 3)$ -string are the same as for the $(1, 1)$ -string and lead to the same degrees of freedom. But since the $(1, 1)$ -string is mapped onto the $(3, 3)$ -string by a symmetry of the theory, namely Θ_1 , the $(3, 3)$ -string is charged under the same factor of the gauge group and does not contribute any further matter. Furthermore, we have to check that the additional identities $\lambda^{(3,3)} = \alpha \gamma_1^{(1)} \lambda^{(1,1)} \gamma_1^{(1)-1}$ ($\alpha = 1$ in the directions $0, \dots, 3$, $\alpha = i$ on T_1 , $\alpha = -i$ on T_2

⁴For notational simplicity in the following we label the eight branes in the arrangement shown in figure 3 with single numbers as given in table 82 of appendix C.

⁵For a summary on projective representations and further references see the appendix of [22].

and $\alpha = 1$ on T_3) are consistent with the degrees of freedom found so far, which is the case, indeed.

Proceeding in a similar manner for the strings $(4, 4), \dots (8, 8)$ we find that the untwisted open string spectrum contains 1V in the adjoint of the gauge group $[Sp(\frac{M}{4})]^4$ and 3C in the $[(A, 1, 1, 1) \oplus (1, A, 1, 1) \oplus (1, 1, A, 1) \oplus (1, 1, 1, A)]$, where A denotes the antisymmetric representation of $Sp(\frac{M}{4})$ and the four factors of the gauge group arise from the strings $(1, 1)$, $(2, 2)$, $(5, 5)$ and $(6, 6)$, respectively. The strings $(3, 3)$, $(4, 4)$, $(7, 7)$ and $(8, 8)$ are related to these strings by the symmetry Θ_1 and do not contribute any further degrees of freedom, as explained above.

We begin the discussion of the open string twisted sectors with the $(1, 2)$ -string. In the sense of the explanation in the beginning of section 3.2, this string forms the Θ_1^3 -twisted sector and the massless states are given by $\psi_{-1/4}^1 |0_{13}, ij\rangle \lambda_{ji}^{(1,2)}$ and $\psi_{-1/4}^2 |0_{13}, ij\rangle \lambda_{ji}^{(1,2)}$, where 0_{13} denotes the Θ_1^3 -twisted NS vacuum. The $(1, 2)$ -string is invariant under Θ_1^2 and Θ_2 . This leads to the constraints on the Chan-Paton factors

$$\lambda^{(1,2)} = \pm \gamma_{20}^{(2)} \lambda^{(1,2)} \gamma_{20}^{(1)-1} = \pm \gamma_{01}^{(2)} \lambda^{(1,2)} \gamma_{01}^{(1)-1}, \quad (43)$$

where the signs are unphysical, since we only know that Θ_1^4 and Θ_2^2 act trivially on the $(1, 2)$ -string and inserting the corresponding γ -matrices yields

$$\lambda^{(1,2)} = \pm M_1 \lambda^{(1,2)} M_1 = \pm M_2 \lambda^{(1,2)} M_2 \quad (44)$$

where the signs are unphysical and thus arbitrary. Calculating the degrees of freedom we obtain that the $(1, 2)$ -string transforms in the bifundamental $(F, F, 1, 1)$ of the gauge group. The strings $(2, 3)$, $(3, 4)$ and $(4, 1)$ are related to the $(1, 2)$ -string by the symmetries $\Omega\mathcal{R}$ and Θ_1 . Since the massless state is twofold degenerated, these strings form 2 scalars in the $(F, F, 1, 1)$. Analogous the strings $(5, 6)$, $(6, 7)$, $(7, 8)$ and $(8, 5)$ form 2 scalars in the $(1, 1, F, F)$. The Θ_1 -twisted sector is equivalent to the Θ_1^3 -twisted sector, which contains the strings $(1, 4)$, $(2, 1)$ etc. Therefore these sectors together yield 2C in the $(F, F, 1, 1) \oplus (1, 1, F, F)$. Since only \mathbb{Z}_4 -twisted intersection points appear, the multiplicity is one.

The $(1, 3)$ -string, i.e. the Θ_1^2 -twisted sector, possesses the additional symmetries $\Omega\mathcal{R}\Theta_1$ and $\Omega\mathcal{R}\Theta_1^3$, which lead to the constraints

$$\lambda^{(1,3)} = - \left(\gamma_{\Omega\mathcal{R}10}^{(3)} \lambda^{(1,3)} \gamma_{\Omega\mathcal{R}10}^{(1)-1} \right)^T = - \left(\gamma_{\Omega\mathcal{R}30}^{(3)} \lambda^{(1,3)} \gamma_{\Omega\mathcal{R}30}^{(1)-1} \right)^T, \quad (45)$$

where the minus signs again arise from the action of $\Omega\mathcal{R}$ on the massless states. Inserting the corresponding γ -matrices yields

$$\lambda^{(1,3)} = -A^T \lambda^{(1,3)T} A = B^T \lambda^{(1,3)T} B. \quad (46)$$

Counting the degrees of freedom, we find that the $(1, 3)$ -string transforms in the $(A, 1, 1, 1)$, where A denotes the antisymmetric representation of $Sp(\frac{M}{4})$. The $(1, 3)$ -string is related to the $(3, 1)$ -string by Θ_1 , just as $(2, 4)$ to $(4, 2)$, $(5, 7)$ to $(7, 5)$ and $(6, 8)$ to $(8, 6)$. The massless states are twofold degenerated and since T_1 and T_2 are \mathbb{Z}_2 -twisted, the multiplicity is 2 in both the **ABA**- and the **ABB**-lattice. Thus the Θ_1^2 -twisted sector contributes 2C in the $[(A, 1, 1, 1) \oplus (1, A, 1, 1) \oplus (1, 1, A, 1) \oplus (1, 1, 1, A)]$ for both lattices.

The $(1, 5)$ -string, i.e. the Θ_2 -twisted sector, is invariant under Θ_1^2 and Θ_2 . Again imposing that the square of these symmetries act trivially on the massless states, we get the constraints

$$\lambda^{(1,5)} = \pm i N_2 \lambda^{(1,5)} M_1 = \pm M_1 \lambda^{(1,3)T} M_2 . \quad (47)$$

This implies that the strings $(1, 5)$, $(5, 1)$, $(3, 7)$ and $(7, 3)$, which are related by $\Omega\mathcal{R}$ and Θ_1 , transform in the $(F, 1, F, 1)$. Analogous the strings $(2, 6)$, $(6, 2)$, $(4, 8)$ and $(8, 4)$ transform in the $(1, F, 1, F)$. The massless states are twofold degenerated. Now the tori T_2 and T_3 are \mathbb{Z}_2 -twisted, thus the multiplicity is 2 in the **ABA**-lattice and 4 in the **ABB**-lattice. Altogether the Θ_2 -twisted sector contributes 2C (4C) in the $(F, 1, F, 1) \oplus (1, F, 1, F)$ for the **ABA** (**ABB**) lattice.

Proceeding in a similar manner for the remaining twisted sectors we obtain the open string twisted massless spectrum shown in table 3. The only peculiarity is the fact that the $\Theta_1\Theta_2$ -twisted sector (and the sector twisted by $\Theta_1^3\Theta_2$) has only one massless ground state.

twist-sector	ABA	ABB	gauge group / matter
untwisted	1V 3C		$[Sp(\frac{M}{4})]^4$ $(A, 1, 1, 1) \oplus (1, A, 1, 1)$ $\oplus (1, 1, A, 1) \oplus (1, 1, 1, A)$
$\Theta_1 + \Theta_1^3$	2C		$(F, F, 1, 1) \oplus (1, 1, F, F)$
Θ_1^2	2C		$(A, 1, 1, 1) \oplus (1, A, 1, 1)$ $\oplus (1, 1, A, 1) \oplus (1, 1, 1, A)$
Θ_2	2C	4C	$(F, 1, F, 1) \oplus (1, F, 1, F)$
$\Theta_1\Theta_2 + \Theta_1^3\Theta_2$	1C	2C	$(F, 1, 1, F) \oplus (1, F, F, 1)$
$\Theta_1^2\Theta_2$	1C	2C	$(F, 1, F, 1) \oplus (1, F, 1, F)$

Table 3: Open string massless spectrum of $\mathbb{Z}_4 \times \mathbb{Z}_2$.

4 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\Omega\mathcal{R}$ -Orientifold

Since the orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is contained as a substructure in the $\mathbb{Z}_4 \times \mathbb{Z}_2$ model discussed above, the calculation is similar and we do not have to go into the details again.

The lattice is described by the shifts $\vec{v} = (1/2, -1/2, 0)$ for the first \mathbb{Z}_2 -factor and $\vec{w} = (0, 1/2, -1/2)$ for the second one and shown in figure 2. The **A**- and **B**-lattice are not interchanged by any insertion from the orientifold group, thus we obtain perturbatively consistent and inequivalent solutions for the lattices **AAA**, **AAB**, **ABB** and **BBB**. The arrangement of rotated D-6-branes we have to introduce is shown in figure 5. For the location of the branes in the fundamental cells, consider figure 4 and discard the diagonal branes.

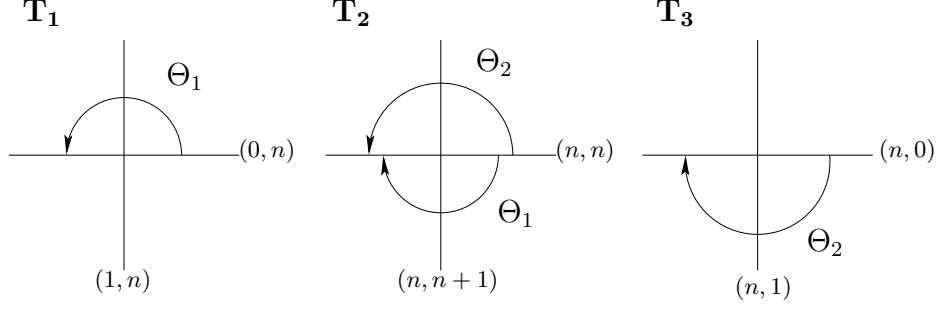


Figure 5: Arrangement of branes and action of the orbifold group for $\mathbb{Z}_2 \times \mathbb{Z}_2$. Inserting $n = 0, 1$ yields the four branes $(i_1, i_2) = (0, 0), (1, 0), (0, 1), (1, 1) \bmod (2, 2)$.

Performing the calculation of the amplitudes, we obtain the untwisted sector tadpole cancellation conditions

$$\begin{aligned}
 \mathbf{AAA}: \quad & [M - 32]^2 = 0, \\
 \mathbf{AAB}: \quad & [M - 16]^2 = 0, \\
 \mathbf{ABB}: \quad & [M - 8]^2 = 0, \\
 \mathbf{BBB}: \quad & [M - 4]^2 = 0,
 \end{aligned} \tag{48}$$

which fix the number M of arrangements of branes shown in figure 5. The remaining tadpole cancellation conditions yield the representation matrices of the orientifold group, which can be read off from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ substructure of tables (82) and (83) in appendix C. Using these matrices, we again solve the constraints arising from the symmetries of the various strings and obtain the massless open string spectrum shown in table (73) in appendix B. The massless closed string spectrum is shown in table (72) in appendix B.

Considering the spectrum we can explicitly verify that for the **AAA** lattice the $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\Omega\mathcal{R}$ -orientifold is T-dual to the model discussed in[11]. Applying T-duality in the directions of the imaginary axes of the compact dimensions transforms $\Omega\mathcal{R}$ to Ω and the four D-6-branes in figure 5 into one D-9-brane and three types of D-5-branes. The **AAB**, **ABB** and **BBB** models are T-dual to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds with discrete B field of rank 2, 4 and 6, respectively, listed in [23]. To our knowledge, orientifolds with discrete B field have been discussed first in [24]. For $\max(N, M) > 2$ the T-duals are asymmetric orientifolds with nonzero B field[25]. Interestingly, for heterotic theories, which are believed to be connected to open string theories, similar observations have been made in[26, 27].

5 The $\mathbb{Z}_6 \times \mathbb{Z}_3$ $\Omega\mathcal{R}$ -Orientifold

The lattice for this model is generated by the shifts $\vec{v} = (1/6, -1/6, 0)$ and $\vec{w} = (0, 1/3, -1/3)$ and shown in figure 6. The **A**- and **B**-lattice are interchanged by $\Omega\mathcal{R}\Theta_1$, thus we get solutions for the lattices **ABA** and **ABB**. To cancel the tadpoles from the closed string we have to introduce the arrangement of 18 rotated branes shown in figure 7 and the number M of arrangements is fixed by the untwisted tadpole cancellation condition

$$[M - 4]^2 = 0, \tag{49}$$

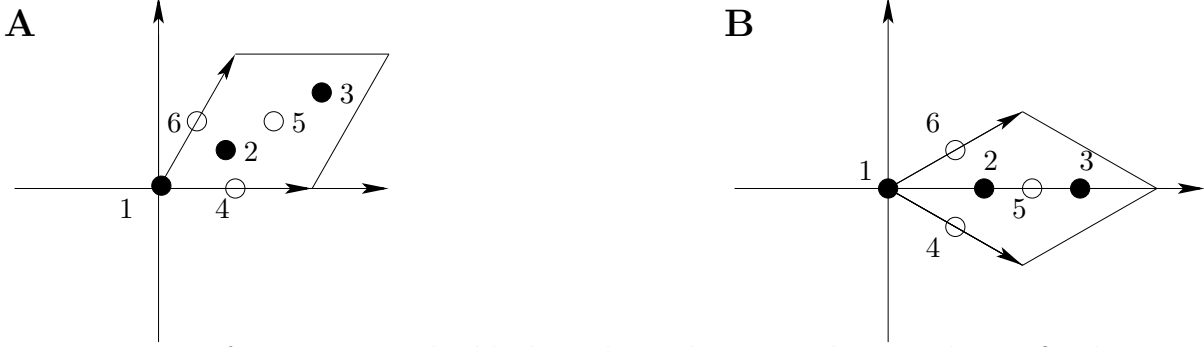


Figure 6: Lattices for $\mathbb{Z}_6 \times \mathbb{Z}_3$. The black circle in the origin denotes the \mathbb{Z}_6 fixed point, the other black (white) circles denote the additional \mathbb{Z}_3 (\mathbb{Z}_2) fixed points.

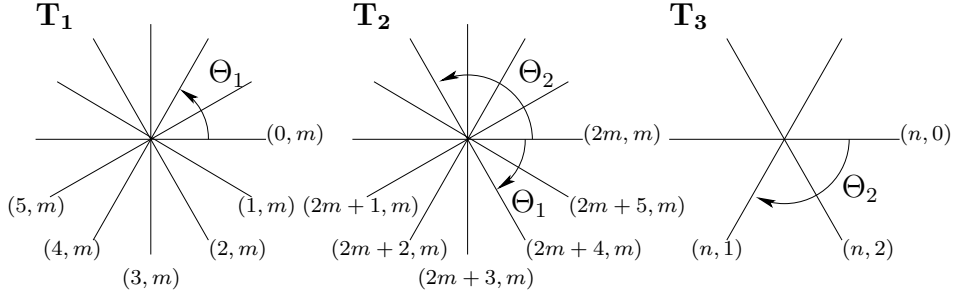


Figure 7: Arrangement of branes and action of the orbifold group for $\mathbb{Z}_6 \times \mathbb{Z}_3$. Inserting $n = 0, \dots, 5$ and $m = 0, 1, 2$ yields 18 branes.

which we obtain for both the lattices **ABA** and **ABB**. Since this model contains less \mathbb{Z}_2 substructure than the two models discussed above, we expect the projective representation of the orientifold group to be less complicated. In fact it turns out that we get by with the γ -matrices of [8], taking into account the relative signs arising from the tadpole cancellation conditions. Calculating the massless open string spectrum shown in table (75) in appendix B we have to take care of the fact that some of the intersection points are not invariant under various insertions. The massless closed string spectrum is shown in table (74) in appendix B.

6 The $\mathbb{Z}_3 \times \mathbb{Z}_3$ $\Omega\mathcal{R}$ -Orientifold

The orbifold shifts are now given by $\vec{v} = (1/3, -1/3, 0)$ and $\vec{w} = (0, 1/3, -1/3)$. The lattice is the same as shown in figure 6, discarding the additional \mathbb{Z}_2 fixed points. The choices **AAA**, **AAB**, **ABB** and **BBB** lead to perturbatively consistent solutions. All these four models yield the untwisted tadpole cancellation condition displayed in equation (49). The calculation of the massless open string spectrum is particularly simple in this case, because the orbifold group contains no \mathbb{Z}_2 -substructure. Therefore we can choose the γ -matrices to be the identity-matrix (up to a phase), while the relative signs are again fixed by the tadpole cancellation conditions. Altogether we obtain the massless spectrum shown in the tables (76) and (77) in appendix B.

7 Conclusions

In this paper we presented a class of orientifolds of type IIA string theory with orbifold group $\mathbb{Z}_N \times \mathbb{Z}_M$. In addition, symmetry under worldsheet parity Ω combined with the reflection \mathcal{R} of three directions was imposed. Further, we considered only cases leading to unbroken $\mathcal{N} = 1$ supersymmetry in the four non compact directions. We found that $(N, M) = (4, 2), (2, 2), (6, 3)$ and $(3, 3)$ are the only perturbatively consistent solutions. For these models we gave the solutions to the tadpole cancellation conditions and the massless spectra. In addition to the universal supergravity fields there are various gauge and matter fields living on D-branes intersecting at angles. The smallest intersection angle is given by $\pi/\max(N, M)$. Explicit results are presented only for the cases where the gauge groups are maximal, i.e. all the D-branes sit at the corresponding orientifold fixed planes. These gauge groups can be Higgsed to smaller groups by moving certain numbers of D-branes off the O-planes.

The type IIA orientifolds considered here can be dualized to type IIB orientifolds by performing T-duality in the directions where \mathcal{R} acts in a non-trivial way. As stated in the end of section 4 the resulting type IIB orientifolds have in most cases a non-trivial discrete B field background. (With B we denote the NSNS antisymmetric tensor.) Constant B field backgrounds have received some attention in the recent past because they can lead to a microscopic description of non-commutative field theories[28–31]. In this context it may be also interesting whether the IIA orientifolds considered here can be modified to include non trivial B field backgrounds. In order to study this question one should investigate whether the tadpole cancellation conditions can be solved by projective representations of the orientifold group which are not equivalent to the ones given here[32, 33]. Note that B_{ij} is quantized only if \mathcal{R} acts with the same sign on x^i and x^j . Otherwise B_{ij} is a modulus and instead G_{ij} (the off-diagonal component of the metric) is quantized[34].

Acknowledgments

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A Computation of one-loop diagrams

In this appendix we will give the details of the computation of the diagrams fig. 1 in the loop channel, i.e. evaluate the expressions (10), (11), and (12). We follow closely the notation of [20]. First, we introduce abbreviations by identifying these expressions with

$$\mathcal{K} = (1-1) 4c \int_0^\infty \frac{dt}{t^3} \left(\frac{1}{4NM} \sum_{n_1, k_1=0}^N \sum_{n_2, k_2=0}^M \mathcal{K}^{(n_1, k_1)(n_2, k_2)} \mathcal{L}_{\mathcal{K}}^{(n_1, k_1)(n_2, k_2)} \right), \quad (50)$$

$$\mathcal{M} = -(1-1) c \int_0^\infty \frac{dt}{t^3} \left(\frac{1}{4NM} \sum_{n_1, k_1=0}^N \sum_{n_2, k_2=0}^M \sum_{(i_1, i_2)=(0,0)}^{(N-1, M-1)} \text{tr} \left(\left(\gamma_{\Omega \mathcal{R} k_1 k_2}^{(i_1, i_2)} \right)^{-1} \left(\gamma_{\Omega \mathcal{R} k_1 k_2}^{(i_1-n_1, i_2-n_2)} \right)^T \right) \right. \\ \left. \mathcal{M}^{(n_1, k_1)(n_2, k_2)} \mathcal{L}_{\mathcal{M}}^{(n_1, k_1)(n_2, k_2)(i_1, i_2)} \right), \quad (51)$$

$$\mathcal{A} = (1-1) c \int_0^\infty \frac{dt}{t^3} \left(\frac{1}{4NM} \sum_{n_1, k_1=0}^N \sum_{n_2, k_2=0}^M \sum_{(i_1, i_2)=(0,0)}^{(N-1, M-1)} \text{tr} \left(\gamma_{k_1 k_2}^{(i_1, i_2)} \right) \text{tr} \left(\left(\gamma_{k_1 k_2}^{(i_1-n_1, i_2-n_2)} \right)^{-1} \right) \right. \\ \left. \mathcal{A}^{(n_1, k_1)(n_2, k_2)} \mathcal{L}_{\mathcal{A}}^{(n_1, k_1)(n_2, k_2)(i_1, i_2)} \right). \quad (52)$$

Let us first explain the meaning of the various symbols in words and later on give the explicit expressions. The letters $\mathcal{K}^{(\cdot)(\cdot)}$, $\mathcal{M}^{(\cdot)(\cdot)}$ and $\mathcal{A}^{(\cdot)(\cdot)}$ stand for oscillator contributions. The upper quadruple index describes the twist sector and the insertion of a $\mathbb{Z}_N \times \mathbb{Z}_M$ element as follows: the contribution corresponds to the $\Theta_1^{n_1} \Theta_2^{n_2}$ twisted sector with a $\Theta_1^{k_1} \Theta_2^{k_2}$ insertion in the trace. (For simplicity, open strings ending on different types of D-branes are called twisted as stated in the text.) The γ 's are the matrix representations of the orientifold group as in [8]. The lower index stands for the corresponding group element, e.g. $\Omega \mathcal{R} k_1 k_2$ corresponds to $\Omega \mathcal{R} \Theta_1^{k_1} \Theta_2^{k_2}$. The upper double index (i_1, i_2) labels the different types of D-6-branes as described in the text. Finally, \mathcal{L} stands for the lattice contribution (i.e. sums over discrete momenta and windings). The indexing is like in the oscillator contributions given above.

A.1 Lattice contributions

The explicit expressions for the lattice contributions are

$$\mathcal{L}_{\mathcal{K}}^{(n_1, k_1)(n_2, k_2)} = \chi_{\mathcal{K}}^{(n_1, k_1)(n_2, k_2)} \text{Tr}_{\text{KK+W}}^{(n_1, n_2)} \left(\Omega \mathcal{R} \Theta_1^{k_1} \Theta_2^{k_2} e^{-2\pi t(L_0 + \bar{L}_0)} \right), \quad (53)$$

$$\mathcal{L}_{\mathcal{M}}^{(n_1, k_1)(n_2, k_2)(i_1, i_2)} = \chi_{\mathcal{M}}^{(n_1, k_1)(n_2, k_2)(i_1, i_2)} \text{Tr}_{\text{KK+W}}^{(i_1, i_2), (i_1-n_1, i_2-n_2)} \left(\Omega \mathcal{R} \Theta_1^{k_1} \Theta_2^{k_2} e^{-2\pi t L_0} \right), \quad (54)$$

$$\mathcal{L}_{\mathcal{A}}^{(n_1, k_1)(n_2, k_2)(i_1, i_2)} = \chi_{\mathcal{A}}^{(n_1, k_1)(n_2, k_2)(i_1, i_2)} \text{Tr}_{\text{KK+W}}^{(i_1, i_2), (i_1-n_1, i_2-n_2)} \left(\Theta_1^{k_1} \Theta_2^{k_2} e^{-2\pi t L_0} \right). \quad (55)$$

In the Klein bottle χ is the number of the corresponding fixed points⁶ whereas in the open string amplitudes χ is the intersection number of the branes involved. (The indexing is analogous to the one described above.) The upper index at Traces gives the twist sector. Sums over windings

⁶In more detail: It is the number of $\Theta_1^{n_1} \Theta_2^{n_2}$ fixed points which are left invariant under $\mathcal{R} \Theta_1^{k_1} \Theta_2^{k_2}$.

and momenta lead to expressions of the form

$$\mathcal{L}[\alpha, \beta] \equiv \left(\sum_{m \in \mathbb{Z}} e^{-\alpha \pi t m^2 / \rho} \right) \left(\sum_{n \in \mathbb{Z}} e^{-\beta \pi t n^2 / \rho} \right), \quad (56)$$

where $\rho = r^2 / \alpha'$. In the tree channel the corresponding function is defined as

$$\tilde{\mathcal{L}}[\alpha, \beta] \equiv \left(\sum_{m \in \mathbb{Z}} e^{-\alpha \pi l m^2 \rho} \right) \left(\sum_{n \in \mathbb{Z}} e^{-\beta \pi l n^2 / \rho} \right). \quad (57)$$

The transformation from the loop channel to the tree channel is performed by Poisson resummation,

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 / t} = \sqrt{t} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}. \quad (58)$$

The exact form of lattice contributions depends on the model and the results are given in the text.

A.2 Oscillator contributions

The general expressions for the oscillator contributions are

$$\mathcal{K}^{(n_1, k_1)(n_2, k_2)} = \text{Tr}_{\text{NSNS}}^{(n_1, n_2)} \left(\Omega \mathcal{R} \Theta_1^{k_1} \Theta_2^{k_2} (-1)^F e^{-2\pi t (L_0 + \bar{L}_0)} \right), \quad (59)$$

$$\mathcal{M}^{(n_1, k_1)(n_2, k_2)} = \text{Tr}_{\text{R}}^{(0, 0)(-n_1, -n_2)} \left(\Omega \mathcal{R} \Theta_1^{k_1} \Theta_2^{k_2} e^{-2\pi t L_0} \right), \quad (60)$$

$$\mathcal{A}^{(n_1, k_1)(n_2, k_2)} = \text{Tr}_{\text{NS}}^{(0, 0)(-n_1, -n_2)} \left(\Theta_1^{k_1} \Theta_2^{k_2} (-1)^F e^{-2\pi t L_0} \right). \quad (61)$$

Because of the $\Omega \mathcal{R}$ insertion in the Klein bottle partition function the expression (59) is actually independent of k_1, k_2 and we define

$$\mathcal{K}^{(n_1, n_2)} \equiv \mathcal{K}^{(n_1, k_1)(n_2, k_2)}. \quad (62)$$

The upper index at the open string amplitudes indicates the boundary conditions, i.e. the computation is done for a string stretching between brane $(0, 0)$ and $(-n_1, -n_2)$. Equivalently, one could trace over open strings stretching between branes (i_1, i_2) and $(i_1 - n_1, i_2 - n_2)$.

The oscillator contributions can be expressed in terms of Jacobi theta functions and the Dedekind eta function,

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (t) = \sum_{n \in \mathbb{Z}} q^{\frac{(n+\alpha)^2}{2}} e^{2\pi i (n+\alpha)\beta}, \quad (63)$$

$$\eta(t) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad (64)$$

with $q = e^{-2\pi t}$. One finds

$$\begin{aligned} \mathcal{K}^{(n_1, n_2)} &= \frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}{\eta^3} \prod_{n_1 v_i + n_2 w_i \notin \mathbb{Z}} \left(\frac{\vartheta \begin{bmatrix} n_1 v_i + n_2 w_i \\ 1/2 \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 + n_1 v_i + n_2 w_i \\ 1/2 \end{bmatrix}} e^{\pi i \langle n_1 v_i + n_2 w_i \rangle} \right) \\ &\quad \prod_{n_1 v_i + n_2 w_i \in \mathbb{Z}} \left(\frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}{\eta^3} \right), \end{aligned} \quad (65)$$

$$\begin{aligned} \mathcal{M}^{(n_1, k_1)(n_2, k_2)} &= \frac{\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}{\eta^3} \prod_{(n_1 v_i + n_2 w_i, k_1 v_i + k_2 w_i) \notin \mathbb{Z}^2} \left(\frac{(-2i)^\delta \vartheta \begin{bmatrix} 1/2 + n_1 v_i + n_2 w_i \\ k_1 v_i + k_2 w_i \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 + n_1 v_i + n_2 w_i \\ 1/2 + k_1 v_i + k_2 w_i \end{bmatrix}} e^{\pi i \langle n_1 v_i + n_2 w_i \rangle} \right) \\ &\quad \prod_{(n_1 v_i + n_2 w_i, k_1 v_i + k_2 w_i) \in \mathbb{Z}^2} \left(\frac{\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}{\eta^3} \right), \end{aligned} \quad (66)$$

$$\begin{aligned} \mathcal{A}^{(n_1, k_1)(n_2, k_2)} &= \frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}{\eta^3} \prod_{(n_1 v_i + n_2 w_i, k_1 v_i + k_2 w_i) \notin \mathbb{Z}^2} \left(\frac{(-2i)^\delta \vartheta \begin{bmatrix} n_1 v_i + n_2 w_i \\ 1/2 + k_1 v_i + k_2 w_i \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 + n_1 v_i + n_2 w_i \\ 1/2 + k_1 v_i + k_2 w_i \end{bmatrix}} e^{\pi i \langle n_1 v_i + n_2 w_i \rangle} \right) \\ &\quad \prod_{(n_1 v_i + n_2 w_i, k_1 v_i + k_2 w_i) \in \mathbb{Z}^2} \left(\frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}{\eta^3} \right). \end{aligned} \quad (67)$$

The arguments in the theta and eta functions are $2t$ in the Klein bottle, $t + \frac{i}{2}$ in the Möbius strip, and t in the annulus. Further, we used the notation[20],

$$\langle x \rangle \equiv x - [x] - \frac{1}{2}, \quad (68)$$

where the brackets on the rhs denote the integer part and

$$\delta = \begin{cases} 1 & \text{if } (n_1 v_i + n_2 w_i, k_1 v_i + k_2 w_i) \in \mathbb{Z} \times \mathbb{Z} + \frac{1}{2} \\ 0 & \text{else} \end{cases} \quad (69)$$

The tree channel expressions $\tilde{\mathcal{K}}^{(\cdot)}$, $\tilde{M}^{(\cdot)}$ and $\tilde{A}^{(\cdot)}$ can be evaluated with the help of the modular transformation properties,

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (1/t) = \sqrt{t} e^{2\pi i \alpha \beta} \vartheta \begin{bmatrix} -\beta \\ \alpha \end{bmatrix} (t), \quad (70)$$

$$\eta(1/t) = \sqrt{t} \eta(t). \quad (71)$$

As usual, there is a subtlety in the Möbius strip. Before performing the modular transformation one writes the theta functions with complex arguments as a product of theta functions with real arguments[8]. Since the calculation is a straightforward modification of the one presented in the appendix of[20] and the formulas are rather lengthy, we do not give the explicit tree channel expressions here.

B Tables of massless spectra

In this appendix we collect tables giving the massless spectra of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ $\Omega\mathcal{R}$ -orientifolds. The spectrum of the $\mathbb{Z}_4 \times \mathbb{Z}_2$ model is given in the text.

B.1 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ model

Closed spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$				
twist-sector	AAA	AAB	ABB	BBB
untwisted	SUGRA + 6C			
θ_1	16C	16C	12C + 4V	10C + 6V
θ_2	16C	12C + 4V	10C + 6V	10C + 6V
$\theta_1\theta_2$	16C	12C + 4V	12C + 4V	10C + 6V

(72)

Open spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$					
twist-sector	AAA	AAB	ABB	BBB	gauge group / matter
untwisted	1V 3C				$[Sp(\frac{M}{4})]^4$ $(A, 1, 1, 1) \oplus (1, A, 1, 1)$ $\oplus (1, 1, A, 1) \oplus (1, 1, 1, A)$
θ_1	1C	1C	2C	4C	$(F, F, 1, 1) \oplus (1, 1, F, F)$
θ_2	1C	2C	4C	4C	$(F, 1, F, 1) \oplus (1, F, 1, F)$
$\theta_1\theta_2$	1C	2C	2C	4C	$(F, 1, 1, F) \oplus (1, F, F, 1)$

(73)

B.2 The $\mathbb{Z}_6 \times \mathbb{Z}_3$ model

Closed spectrum of $\mathbb{Z}_6 \times \mathbb{Z}_3$		
twist-sector	ABA	ABB
untwisted	SUGRA + 3C	
$\theta_1 + \theta_1^5$	2C	2C
$\theta_1^2 + \theta_1^4$	8C + 2V	8C + 2V
θ_1^3	5C + 1V	5C + 1V
$\theta_2 + \theta_2^2$	8C + 4V	12C
$\theta_1\theta_2 + \theta_1^5\theta_2^2$	2C + 1V	3C
$\theta_1^2\theta_2 + \theta_1^4\theta_2^2$	8C + 4V	12C
$\theta_1^3\theta_2 + \theta_1^3\theta_2^2$	4C + 2V	6C
$\theta_1^4\theta_2 + \theta_1^2\theta_2^2$	9C + 6V	12C + 3V
$\theta_1^5\theta_2 + \theta_1\theta_2^2$	4C + 2V	6C

(74)

Open spectrum of $\mathbb{Z}_6 \times \mathbb{Z}_3$			
twist-sector	ABA	ABB	representation of $U(2) \times U(2)$
untwisted	1V		$(\mathbf{4}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{4})$
	1C		$(\mathbf{4}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{4})$
	4C		$(\mathbf{1}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{1})$
$\theta_1 + \theta_1^5$	4C		$(\mathbf{2}, \bar{\mathbf{2}}) \oplus (\bar{\mathbf{2}}, \mathbf{2})$
$\theta_1^2 + \theta_1^4$	2C		$(\mathbf{4}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{4})$
	8C		$(\mathbf{1}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{1})$
θ_1^3	4C		$(\mathbf{2}, \bar{\mathbf{2}}) \oplus (\bar{\mathbf{2}}, \mathbf{2})$
$\theta_2 + \theta_2^2$	2C	6C	$(\mathbf{4}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{4})$
	4C	12C	$(\mathbf{1}, \mathbf{1}_0)$
	2C	6C	$(\mathbf{4}, \mathbf{1}_0)$
$\theta_1\theta_2 + \theta_1^5\theta_2^2$	2C	6C	$(\mathbf{2}, \bar{\mathbf{2}}) \oplus (\bar{\mathbf{2}}, \mathbf{2})$
$\theta_1^2\theta_2 + \theta_1^4\theta_2^2$	2C	6C	$(\mathbf{1}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{1})$
	4C	12C	$(\mathbf{1}_0, \mathbf{1})$
	2C	6C	$(\mathbf{1}_0, \mathbf{4})$
$\theta_1^3\theta_2 + \theta_1^3\theta_2^2$	4C	12C	$(\mathbf{2}, \bar{\mathbf{2}}) \oplus (\bar{\mathbf{2}}, \mathbf{2})$
$\theta_1^4\theta_2 + \theta_1^2\theta_2^2$	2C	6C	$(\mathbf{4}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{4})$
	2C	6C	$(\mathbf{3}, \mathbf{1}_0) \oplus (\mathbf{1}_0, \mathbf{3})$
$\theta_1^5\theta_2 + \theta_1\theta_2^2$	4C	12C	$(\mathbf{2}, \bar{\mathbf{2}}) \oplus (\bar{\mathbf{2}}, \mathbf{2})$

(75)

B.3 The $\mathbb{Z}_3 \times \mathbb{Z}_3$ model

Closed spectrum of $\mathbb{Z}_3 \times \mathbb{Z}_3$				
twist-sector	AAA	AAB	ABB	BBB
untwisted	SUGRA + 3C			
$\theta_1 + \theta_1^2$	10C + 8V	10C + 8V	12C + 6V	18C
$\theta_2 + \theta_2^2$	10C + 8V	12C + 6V	18C	18C
$\theta_1\theta_2 + \theta_1^2\theta_2^2$	10C + 8V	12C + 6V	12C + 6V	18C
$\theta_1^2\theta_2 + \theta_1\theta_2^2$	14C + 13V	15C + 12V	18C + 9V	27C

(76)

Open spectrum of $\mathbb{Z}_3 \times \mathbb{Z}_3$					
twist-sector	AAA	AAB	ABB	BBB	rep. of $SO(4)$
untwisted	1V				6
	3C				6
$\theta_1 + \theta_1^2$	2C	2C	6C	18C	6
$\theta_2 + \theta_2^2$	2C	6C	18C	18C	6
$\theta_1\theta_2 + \theta_1^2\theta_2^2$	2C	6C	6C	18C	6
$\theta_1^2\theta_2 + \theta_1\theta_2^2$	1C	3C	9C	27C	10

(77)

C Projective representations of $\Omega\mathcal{R} \times \mathbb{Z}_4 \times \mathbb{Z}_2$

In this appendix we give the explicit expressions for the projective representation used in the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold in section 3. The projective representation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subsector is chosen to be equivalent to the one presented in [11]. M_i , N_i and D are as defined in [11],

$$\begin{aligned}
M_i &= \left\{ \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \right\}, \\
D &= \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \\
N_i \equiv DM_i &= \left\{ \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \right\},
\end{aligned}
\tag{78}$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and the matrices fulfill $M_i^2 = N_1^2 = -D^2 = -N_2^2 = -N_3^2 = -\mathbb{1}$. In addition we use the notation

$$a = \frac{1}{2}(\mathbb{1} - \sigma_2), \quad b = -ia^T \quad (79)$$

to define

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, & A^{-1} &= A, \\ B &= \begin{pmatrix} -b & a \\ -a & -b \end{pmatrix}, & B^{-1} &= -B, \\ C &= i \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, & C^{-1} &= i \begin{pmatrix} -a & b \\ b & a \end{pmatrix}, \\ E &= i \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}, & E^{-1} &= i \begin{pmatrix} b & a \\ a & -b \end{pmatrix}. \end{aligned} \quad (80)$$

The latter satisfy $C^T = iE$. Furthermore we need the set of matrices

$$\begin{aligned} F &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ i & -i \end{pmatrix}, \\ G &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -i & i \end{pmatrix}, \\ H &= \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ \mathbb{1} & \mathbb{1} \end{pmatrix}, \\ K &= \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ -\mathbb{1} & -\mathbb{1} \end{pmatrix}, \end{aligned} \quad (81)$$

which satisfy $F^{-T} = G$ and $H^{-T} = -K$. The representation matrices of the orientifold group $\Omega\mathcal{R} \times \mathbb{Z}_4 \times \mathbb{Z}_2$ are listed in table (82). In the first column, we list the systematic numbering of the branes as explained in section 3.2, and in the second column we give the simplified numbering used in section 3.5. All relevant signs are listed explicitly whereas the others are arbitrary.

The representation matrices of the orbifold group $\mathbb{Z}_4 \times \mathbb{Z}_2$ are obtained as explained in equation (40) of section 3.5. They are listed in table (83) up to an irrelevant phase.

Representation-matrices of $\Omega\mathcal{R} \times \mathbb{Z}_4 \times \mathbb{Z}_2$									
brane		$\gamma_{\Omega\mathcal{R}}$	$\gamma_{\Omega\mathcal{R}1}$	$\gamma_{\Omega\mathcal{R}1^2}$	$\gamma_{\Omega\mathcal{R}1^3}$	$\gamma_{\Omega\mathcal{R}2}$	$\gamma_{\Omega\mathcal{R}12}$	$\gamma_{\Omega\mathcal{R}1^22}$	$\gamma_{\Omega\mathcal{R}1^32}$
(0,0)	1	\mathbb{I}	$+A$	M_1	$+B$	M_2	$+C$	M_3	$+E$
(1,0)	2	$+B$	\mathbb{I}	$+A$	M_1	$+E$	M_2	$+C$	M_3
(2,0)	3	N_1	$+A^T$	D	$+B^T$	M_2	$-C^T$	M_3	$+E^T$
(3,0)	4	$+B^T$	N_1	$+A^T$	D	$+E^T$	M_2	$-C^T$	M_3
(0,1)	5	N_1	$+F$	M_3	$+G$	D	$+H$	M_2	$+K$
(1,1)	6	$+G$	N_1	$+F$	M_3	$+K$	D	$+H$	M_2
(2,1)	7	N_1	$-F^T$	M_2	$+G^T$	M_3	$+H^T$	D	$+K^T$
(3,1)	8	$+G^T$	N_1	$-F^T$	M_2	$+K^T$	M_3	$+H^T$	D

(82)

Representation-matrices of $\mathbb{Z}_4 \times \mathbb{Z}_2$								
brane		γ_1	γ_{1^2}	γ_{1^3}	γ_2	γ_{12}	γ_{1^22}	γ_{1^32}
(0,0)	1	B^{-T}	M_1	A^{-T}	M_2	E^{-T}	M_3	C^{-T}
(1,0)	2	B^{-T}	M_1	A^{-T}	M_2	E^{-T}	M_3	C^{-T}
(2,0)	3	A^T	M_1	B^T	N_3	C^T	N_2	E^T
(3,0)	4	A^T	M_1	B^T	N_3	C^T	N_2	E^T
(0,1)	5	$G^{-T}N_1$	N_2	$F^{-T}N_1$	M_1	$K^{-T}N_1$	N_3	$H^{-T}N_1$
(1,1)	6	$G^{-T}N_1$	N_2	$F^{-T}N_1$	M_1	$K^{-T}N_1$	N_3	$H^{-T}N_1$
(2,1)	7	N_1F^T	N_3	N_1G^T	N_2	N_1H^T	M_1	N_1K^T
(3,1)	8	N_1F^T	N_3	N_1G^T	N_2	N_1H^T	M_1	N_1K^T

(83)

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